Matte, eng.

15-4

Partial Derivatives 767-770

Read subheadings

Suppose that $y = f(x)$ is a function of *one* real variable. Its first derivative

$$\frac{dy}{dx} = D_x f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{1}$$

can be interpreted as the instantaneous rate of change of y with respect to x. For a function $z = f(x, y)$ of two variables, we need a similar understanding of the rate at which z changes as x and y vary (either singly or simultaneously). To reach this more complicated concept, we adopt a "divide and conquer" strategy.

First we hold y fixed and let x vary. The rate of change of z with respect to x is then denoted by $\partial z/\partial x$ and has the value

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}. \tag{2}$$

The value of this limit (assuming it exists) is called the **partial derivative of f with respect to x**. In like manner, we may hold x fixed and let y vary. The rate of change of z with respect to y is then the **partial derivative of f with** respect to y, defined to be

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k} \tag{3}$$

for all (x, y) for which this limit exists. Note the symbol ∂ that is used instead of d to denote the partial derivatives of a function of two variables. A function of three or more independent variables has a partial derivative (defined similarly) with respect to each of its independent variables. Some other commonly used notations for partial derivatives are

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x(x, y) = D_x f(x, y) = D_1 f(x, y), \tag{4}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y(x, y) = D_y f(x, y) = D_2 f(x, y) \tag{5}$$

Note that if the symbol y in Equation (2) is deleted, the result is the limit of Equation (1). This means that $\partial z/\partial x$ can be calculated as an "ordinary" derivative with respect to x, simply regarding y as a constant during the process of differentiation. Similarly, we can compute $\partial z/\partial y$ as an ordinary derivative, thinking of y as the *only* variable and treating x as a constant during the computation.

EXAMPLE 1 Compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of the function $f(x, y) = x^2 + 2xy^2 - y^3$.

Solution To compute the partial of f with respect to x, we regard y as a constant. Then we differentiate normally and find that

$$\frac{\partial f}{\partial x} = 2x + 2y^2$$

When we regard x as a constant and differentiate with respect to y, we find that

$$\frac{\partial f}{\partial y} = 4xy - 3y^2$$

To get an intuitive feel for the meaning of partial derivatives, we can think of $f(x, y)$ as the temperature at the point (x, y). Then $f_x(x, y)$ is the instantaneous rate of change of temperature at (x, y) per unit increase in x (with y held constant), while $f_y(x, y)$ is the rate of change of temperature per unit increase in y (with x held constant). For example, with the temperature function $f(x, y) = x^2 + 2xy^2 - y^3$ of Example 1, the rate of change of temperature at the point $(1, -1)$ is $+4°$ per unit distance in the positive x-direction and $-7°$ per unit distance in the positive y-direction.

EXAMPLE 2 Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = (x^2 + y^2)e^{-xy}$

Solution Because $\partial z/\partial x$ is calculated as if it were an ordinary derivative with respect to x while y is held constant, we can use the product rule. This gives

$$\begin{aligned}\frac{\partial z}{\partial x} &= (2x)(e^{-xy}) + (x^2 + y^2)(-ye^{-xy}) \\ &= (2x - x^2y - y^3)e^{-xy}\end{aligned}$$

Because x and y appear symmetrically in the expression $z = (x^2 + y^2)e^{-xy}$, we get $\partial z/\partial y$ when we interchange x and y in the expression for $\partial z/\partial x$:

$$\frac{\partial z}{\partial y} = (2y - xy^2 - x^3)e^{-xy}$$

You should check this result by differentiating z with respect to y.

EXAMPLE 3 The volume V (in cubic centimeters) of 1 mole of an ideal gas is given by

$$V = \frac{(82.06)T}{P},$$

where P is the pressure (in atmospheres) and T is the absolute temperature (in degrees Kelvin (°K), where °K = °C + 273). Find the rates of change of the volume of 1 mol of an ideal gas with respect to pressure and with respect to temperature with $T = 300$°K and $P = 5$ atm.

Solution The partial derivatives of V with respect to its two variables are

$$\frac{\partial V}{\partial P} = -\frac{(82.06)T}{P^2} \quad \text{and} \quad \frac{\partial V}{\partial T} = \frac{82.06}{P}.$$

With $T = 300$°K and $P = 5$ atm, we have the two values $\partial V/\partial P = -984.72$ cm^3/atm and $\partial V/\partial T = 16.41$ cm^3/°K. These partial derivatives allow us to estimate the effect of a change in temperature or in pressure on the volume V of gas in question, as follows. We are given $T = 300$°K and $P = 5$ atm, so the volume of gas we are dealing with is $V = (82.06)(300)/5 = 4923.60$ cm^3. We would expect an increase in pressure of one atmosphere (with temperature held constant) to decrease the volume of gas by roughly 1 liter (1000 cm^3). An increase in temperature of 1°K (or 1°C) would, with pressure held constant, increase the volume by about 16 cm^3.

15.4.1 GEOMETRIC INTERPRETATION OF PARTIAL DERIVATIVES (769) - 770

The partial derivatives f_x and f_y are the slopes of tangent lines to certain curves on the surface $z = f(x, y)$. Consider the point $P(a, b, f(a, b))$ on this surface. It lies directly above the point $Q(a, b, 0)$ in the xy-plane, as shown in Fig. 15.40. The vertical plane $y = b$ parallel to the xz-plane intersects the surface in the curve $z = f(x, b)$ through the point P. Along this curve, x varies

15.40 Geometric interpretation of $f_x(a, b)$ as the slope of a tangent to a curve

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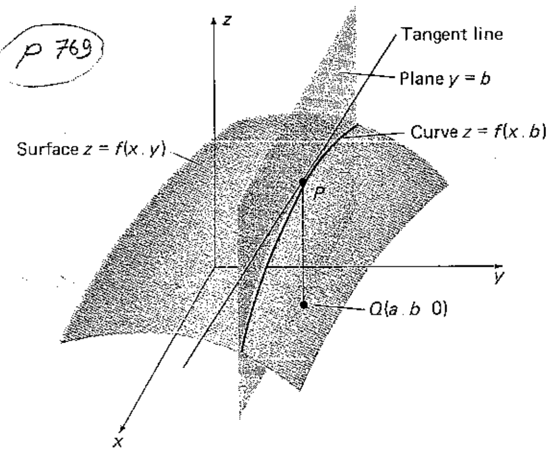

but y is constant; y is always equal to b because the curve lies in the vertical plane with equation $y = b$

Let us project this curve and its tangent line at P into the xz-plane. We get exactly the same curve and tangent line, for this is a normal projection. But because we are now working in the xz-plane, we can actually "forget" the presence of y. In effect, as indicated in Fig 15.41, we are dealing with z as a function of the *single* variable x, and the projected curve is the graph of this function. Hence the slope of the tangent line to the original curve at the point $P(a, b, f(a, b))$ is equal to the slope of the tangent line of Fig 15.41. But by familiar single-variable calculus, this slope is

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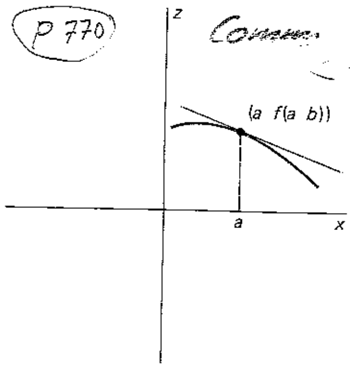


15.41 The projection of the curve and tangent line of Fig. 15.40 into the xz-plane

$$\lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} = f_x(a, b).$$

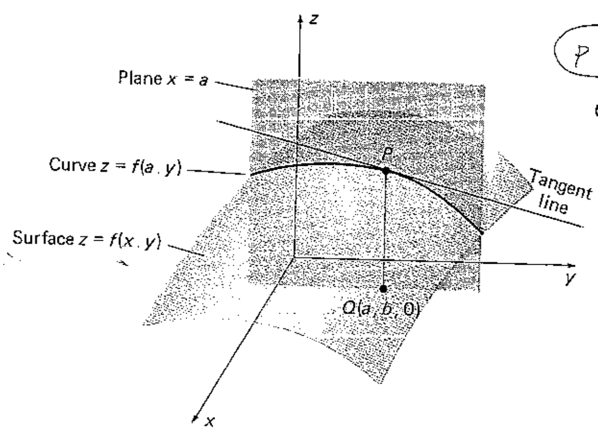
Thus $f_x(a, b)$ is the slope of the tangent line at P to the curve formed by intersecting the plane $y = b$ parallel to the xz-plane with the surface $z = f(x, y)$

We proceed in much the same way to get a geometric interpretation of the partial derivative of f with respect to y. As Fig 15.42 suggests, *$f_y(a, b)$ is the slope of a line tangent to the curve of intersection of the surface $z = f(x, y)$ with the plane $x = a$ parallel to the yz-plane* fig 15.42 below

The two tangent lines we have just found determine a unique plane through the point $P(a, b, f(a, b))$. In Section 15-7 we will see that if the partial derivatives f_x and f_y are continuous functions of x and y, then this plane contains the tangent line at P to *every* smooth curve on the surface $z = f(x, y)$ that passes through P. This plane is therefore (by definition) the tangent plane to the surface at P.

> ***Definition*** *Tangent Plane to $z = f(x, y)$*
>
> Suppose that the function $f(x, y)$ has continuous partial derivatives on a rectangle in the xy-plane containing (a, b) in its interior. Then the **tangent plane** to the surface $z = f(x, y)$ at the point $P(a, b, f(a, b))$ is the plane through P that contains the tangent lines to the two curves
>
> $$z = f(x, b), \qquad y = b \tag{6}$$
>
> and
>
> $$z = f(a, y), \qquad x = a. \tag{7}$$

15.42 The partial derivative f_y is also the slope of a line tangent to a curve in the surface $z = f(x, y)$



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